

## ELEMENTARY EQUIVALENCE OF LINEAR GROUPS OVER GRADED RINGS WITH FINITE NUMBER OF CENTRAL IDEMPOTENTS

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*Dedicated to the memory of Professor Marc Krasner*

ABSTRACT. In this paper we prove the criterion of elementary equivalence of linear groups over graded rings with finite number of central idempotents from the 0-component, when grading is partially included in the group language.

### 1. INTRODUCTION

In this paper we study the criterion of elementary equivalence of linear groups over graded rings with the condition that grading is *partially* used in the group (see below all detailed definitions).

The first result in elementary classification of linear groups was obtained by A.I. Maltsev in the paper [1]. He proved the following

**Theorem 1.** *A group  $\mathcal{G}_m(\mathbb{K}_1)$  is elementary equivalent to  $\mathcal{G}_n(\mathbb{K}_2)$  ( $\mathcal{G} = \text{GL}_n, \text{PGL}_n, \text{SL}_n, \text{PSL}_n, m, n \geq 3, \mathbb{K}_1, \mathbb{K}_2$  are infinite fields) if and only if  $m = n$  and  $\mathbb{K}_1 \cong \mathbb{K}_2$ .*

Then K.I. Beidar and A.V. Mikhalev found a general approach to the problems of elementary equivalence of some algebraic structures (see [2]). Taking into account some results of linear groups theory over rings (see [3], [4], [5]) they obtained relatively simple proofs of Maltsev-type theorems in quite general situations (for linear groups over prime rings and skewfields,

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2010 *Mathematics Subject Classification.* 03C60.

*Key words and phrases.* Elementary equivalence, linear groups, graded rings, central idempotents.

This paper was presented at the International Scientific Conference *Graded structures in algebra and their applications*, dedicated to the memory of Prof. Marc Krasner, IUC-Dubrovnik, Croatia, September, 22-24, 2016.

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multiplicative semigroups of rings, lattices of submodules and so on). In [2] they obtained the following two theorems:

**Theorem 2.** *Let  $R$  and  $S$  be skewfields (of characteristics  $\neq 2$ ) and  $m, n \geq 3$  ( $m, n \geq 2$ ). Then  $\mathrm{GL}_m(R) \cong \mathrm{GL}_n(S)$  if and only if  $n = m$  and either  $R \cong S$ , or  $R \cong S^{op}$ .*

**Theorem 3.** *Let  $R$  and  $S$  be associative prime rings with  $1$  ( $1/2$ ),  $m, n \geq 4$  ( $m, n \geq 3$ ). Then  $\mathrm{GL}_m(R) \cong \mathrm{GL}_n(S)$  if and only if either  $M_m(R) \cong M_n(S)$ , or  $M_m(R) \cong M_n(S)^{op}$ .*

By similar methods in the paper [6], the following generalization of the previous theorem was proved:

**Theorem 4.** *Let  $R$  and  $S$  be associative rings with  $1$  ( $1/2$ ) with finite number of central idempotents and  $m, n \geq 4$  ( $m, n \geq 3$ ). Then  $\mathrm{GL}_m(R) \cong \mathrm{GL}_n(S)$  if and only if there exist such central idempotents  $e \in R$  and  $f \in S$  that  $eM_m(R) \cong fM_n(S)$  and  $(1 - e)M_m(R) \cong (1 - f)M_n(S)^{op}$ .*

But in the case if there is infinite number of central idempotents, similar theorem can not be proved. Of course the analogue of this theorem holds for different particular cases of rings. For example in the paper [6] the criterion for linear groups over Boolean rings was obtained:

**Theorem 5.** *Let  $\mathcal{B}_1$  and  $\mathcal{B}_2$  be Boolean rings,  $n, m$  be natural numbers. Linear groups  $\mathrm{GL}_n(\mathcal{B}_1)$  and  $\mathrm{GL}_n(\mathcal{B}_2)$  are elementary equivalent if and only if  $n = m$  and the rings  $\mathcal{B}_1$  and  $\mathcal{B}_2$  are elementary equivalent.*

Similar theorems were proved not only for classical linear groups  $\mathrm{GL}$ ,  $\mathrm{PGL}$ ,  $\mathrm{SL}$ ,  $\mathrm{PSL}$ , but also for unitary linear groups over fields, skewfields and rings with involutions (see [7], [8]), for Chevalley groups over fields ([9]) and local rings ([10]), and for different other derivative structures.

The interesting problem is to extend the criteria of elementary equivalence for linear groups over different classes of graded rings. The main difficulty which appears here is the fact that grading can be extended automatically from rings and corresponding matrix rings to linear groups. Therefore it is necessary to determine specially what we mean by the notion of *graded* linear groups.

As usual the problem of classification up to elementary equivalence of some derivative structures makes sense after classification of these structures up to isomorphism or even full description of an arbitrary isomorphism between the structures under consideration.

In 2010 A.S. Atkarskaya, E.I. Bunina and A.V. Mikhalev in [11] consider namely the following problem: description of isomorphisms between linear groups over graded (by an Abelian group) rings, the isomorphisms under consideration in some sense *respect* grading.

In the paper [11] the following theorem was proved (detailed definitions are given in the next section):

**Theorem 6.** *Let  $G$  be an Abelian group,*

$$R = \bigoplus_{g \in G} R_g, \quad S = \bigoplus_{g \in G} S_g$$

*be associative graded rings with 1,  $M_n(R)$ ,  $M_m(S)$  be the corresponding graded matrix rings,  $n \geq 4$ ,  $m \geq 4$ , and  $\varphi : \text{GL}_n(R) \rightarrow \text{GL}_m(S)$  be a group isomorphism respecting grading. Suppose that the isomorphism  $\varphi^{-1}$  also respects grading.*

*Then there exist central idempotents  $e$  and  $f$  of the rings  $M_n(R)$  and  $M_m(S)$  respectively,  $e \in M_n(R)_0$ ,  $f \in M_m(S)_0$ , a ring isomorphism*

$$\theta_1 : eM_n(R) \rightarrow fM_m(S)$$

*and a ring anti-isomorphism*

$$\theta_2 : (1 - e)M_n(R) \rightarrow (1 - f)M_m(S),$$

*respecting grading and such that*

$$\varphi(A) = \theta_1(eA) + \theta_2((1 - e)A^{-1})$$

*for all  $A \in E_n(R)$ .*

Therefore in this paper we prove the following theorem:

**Theorem 7.** *Let  $G$  be a commutative group,*

$$R = \bigoplus_{g \in G} R_g, \quad S = \bigoplus_{g \in G} S_g$$

*be associative graded rings with 1,  $M_n(R)$ ,  $M_m(S)$  be the corresponding graded matrix rings,  $n \geq 4$ ,  $m \geq 4$ . Suppose that also both rings  $R$ ,  $S$  contain only finite number of central idempotents from the components  $R_0$  and  $S_0$ , respectively.*

*Then elementary equivalence of the groups  $\text{GL}_n(R)$  and  $\text{GL}_m(S)$  in the language respecting grading (see §3) is equivalent to the existence of central idempotents  $e$  and  $f$  of the rings  $M_n(R)$  and  $M_m(S)$  respectively,  $e \in M_n(R)_0$ ,  $f \in M_m(S)_0$ , such that*

$$eM_n(R) \equiv_{\text{gr}} fM_m(S), \quad (1 - e)M_n(R) \equiv_{\text{gr}} (1 - f)M_m(S)^{op}$$

*as graded rings.*

## 2. LINEAR GROUPS OVER GRADED RINGS

Main definitions concerning graded rings and their endomorphisms are taken from [12], [13].

All rings are supposed to be associative with unit. Suppose that we have some group  $G$ .

**Definition 1.** A ring  $R$  is called  $G$ -graded (or *graded by the group  $G$* ), if  $R = \bigoplus_{g \in G} R_g$ , where  $\{R_g \mid g \in G\}$  is a system of additive subgroups of the ring  $R$  and  $R_g R_h \subseteq R_{gh}$  for all  $g, h \in G$ . Moreover, if  $R_s R_h = R_{sh}$  for all  $s, h \in G$ , then the ring is called *strongly graded*.

**Definition 2.** Two  $G$ -graded rings  $R$  and  $S$  are called *isomorphic* (or *gr-isomorphic*) if there exists a ring isomorphism  $f : R \rightarrow S$  such that  $f(R_g) \cong S_g$  for all  $g \in G$ .

**Definition 3.** A right  $R$ -module  $M$  is called  $G$ -graded if  $M = \bigoplus_{g \in G} M_g$ , where  $\{M_g \mid g \in G\}$  is a system of additive subgroups in  $M$  such that  $M_h R_g \subseteq M_{hg}$  for all  $h, g \in G$ .

**Definition 4.** An  $R$ -linear mapping  $f : M \rightarrow N$  of right  $G$ -graded  $R$ -modules is called *a graded morphism of degree  $g$*  if  $f(M_h) \subseteq N_{gh}$  for all  $h \in G$ . The set of graded morphisms of degree  $g$  is the subgroup  $\text{HOM}_R(M, N)_g$  of the group  $\text{Hom}_R(M, N)$ .

**Definition 5.** Let  $\text{END}_R(M) := \bigoplus_{g \in G} \text{HOM}_R(M, M)_g$ . This graded ring is called the *graded endomorphism ring* of the graded  $R$ -module  $M$ .

**Definition 6.** A graded right  $R$ -module  $M$  is called *gr-free* if it has a basis consisting of homogenous elements.

It is known (see, for example, [13]), that if  $R = \bigoplus_{g \in G} R_g$  is an associative graded ring with 1,  $M$  is a finitely generated *gr-free* right  $R$ -module with a basis consisting of homogeneous elements  $v_1, v_2, \dots, v_n$ , where  $v_i \in M_{g_i}$  ( $i = 1, \dots, n$ ), then the graded endomorphism ring  $\text{END}_R(M)$  is isomorphic to the graded matrix ring

$$M_n(R)(g_1, \dots, g_n) = \bigoplus_{h \in G} M_n(R)_h(g_1, \dots, g_n),$$

where

$$M_n(R)_h(g_1, \dots, g_n) = \begin{pmatrix} R_{g_1^{-1}hg_1} & R_{g_1^{-1}hg_2} & \dots & R_{g_1^{-1}hg_n} \\ R_{g_2^{-1}hg_1} & R_{g_2^{-1}hg_2} & \dots & R_{g_2^{-1}hg_n} \\ \vdots & \vdots & \ddots & \vdots \\ R_{g_n^{-1}hg_1} & R_{g_n^{-1}hg_2} & \dots & R_{g_n^{-1}hg_n} \end{pmatrix}.$$

**Definition 7.** Let  $R = \bigoplus_{g \in G} R_g$  and  $S = \bigoplus_{g \in G} S_g$  be associative graded rings with 1,  $M_n(R)$ ,  $M_n(S)$  be graded matrix rings. A group isomorphism

$$\varphi : GL_n(R) \longrightarrow GL_m(S)$$

is called an *isomorphism respecting grading*, if for all  $g \in G$  and for all  $A \in GL_n(R)$

$$A - E \in M_n(R)_g \implies \varphi(A) - E \in M_m(S)_g.$$

**Definition 8.** By  $E_n(R)$  we denote the subgroup of  $GL_n(R)$  generated by all matrices  $E + re_{ij}$ ,  $1 \leq i, j \leq n$ ,  $i \neq j$ ,  $r \in R$ .

The following theorem was proved by I.Z. Golubchik [5], who generalized the results of I.Z. Golubchik and A.V. Mikhalev [3]:

**Theorem 8.** *Let  $R$  and  $S$  be associative rings with unit,  $n \geq 4$ ,  $m \geq 2$  and  $\varphi : GL_n(R) \longrightarrow GL_m(S)$  be a group isomorphism. Then there exist central idempotents  $e$  and  $f$  of the rings  $M_n(R)$  and  $M_m(S)$  respectively, a ring isomorphism*

$$\theta_1 : eM_n(R) \rightarrow fM_m(S)$$

and a ring antiisomorphism

$$\theta_2 : (1 - e)M_n(R) \rightarrow (1 - f)M_m(S)$$

such that

$$\varphi(A) = \theta_1(eA) + \theta_2((1 - e)A^{-1})$$

for all  $A \in E_n(R)$ .

Namely, with the help of this theorem, Theorem 6 in the paper [11] was proved. We will use it below to prove our main theorem.

For any commutative group we shall denote an operation by  $+$  and we shall denote a unit element by 0.

### 3. PROOF OF THE MAIN THEOREM

As we formulated already in Introduction our goal is to prove the analogue of theorem of elementary equivalence of linear groups, switching from usual associative rings to graded rings and using Theorem 6.

Since in Theorem 6 the important moment was in definition of isomorphism respecting grading, then for elementary equivalence it is necessary to define how grading is included in the first order language of our linear group.

Let us start with this notion.

For a linear group  $GL_n(R)$ , where  $R$  is a ring graded by an Abelian group  $G$ , the group  $GL_n(R)$  itself is embedded into the graded matrix ring  $M_n(R)$ , we will consider the model

$$\langle M \mid \circ, {}^{-1}, 1; I_g (g \in G) \rangle,$$

where  $M$  is our basic model (i. e., the group  $\text{GL}_n(R)$ ),  $\circ$  is the multiplication in  $M$ ,  $^{-1}$  is taking an inverse in  $M$ ,  $1$  is the unity of  $M$ ,  $I_g$  ( $g \in G$ ) are one-placed predicates (one for each  $g \in G$ ), which are true for an element  $x \in M$  if and only if

$$x - 1 \in M_n(R)_g.$$

Let us call the introduced first order language *group language with grading*, elementary equivalence of two models of this language will be denoted by  $\equiv_{\text{gr}}$ .

For the proof we need the Keisler–Shelah Isomorphism theorem:

**Theorem 9** ([14], [15]). *Two models  $\mathcal{U}_1$  and  $\mathcal{U}_2$  of the same first order language are elementary equivalent if and only if there exists an ultrafilter  $\mathcal{F}$  such that*

$$\prod_{\mathcal{F}} \mathcal{U}_1 \cong \prod_{\mathcal{F}} \mathcal{U}_2.$$

Let us now prove the main theorem.

*Proof.* Suppose that two groups  $\text{GL}_n(R)$  and  $\text{GL}_m(S)$  over rings, graded by an Abelian group  $G$  and having only finite number of central idempotents from the 0-component, are elementary equivalent in the group language with grading:

$$\text{GL}_n(R) \equiv_{\text{gr}} \text{GL}_m(S).$$

By Keisler–Shelah Isomorphism theorem this is equivalent to the fact that their ultrapowers by ultrafilter  $\mathcal{F}$  are isomorphic:

$$\prod_{\mathcal{F}} \text{GL}_n(R) \cong \prod_{\mathcal{F}} \text{GL}_m(S).$$

Since any isomorphism preserves all constants, functions and predicates of the corresponding language, then for our isomorphism

$$\Phi : \prod_{\mathcal{F}} \text{GL}_n(R) \rightarrow \prod_{\mathcal{F}} \text{GL}_m(S)$$

$I_g(X)$  implies  $I_g(\Phi(X))$ .

If  $\mathcal{F}$  is an ultrafilter over  $J$ , then for  $X = (x_j \in \text{GL}_n(R) \mid j \in J)$  the condition  $I_g(X)$  is equivalent to

$$\{j \in J \mid x_j - E \in M_n(R)_g\} \in \mathcal{F}.$$

Presenting the element  $X$  as a matrix having on its  $k, l$ -th place

$$X^{k,l} = (x_j^{k,l} \mid j \in J),$$

the condition  $I_g(X)$  is equivalent to

$$\{j \in J \mid x_j^{k,l} - \delta_{k,l} \in R_{g_k^{-1}ggi}\} \in \mathcal{F} \text{ for all } k, l = 1, \dots, n$$

in the basis of elements belonging to the components  $g_1, \dots, g_n$ .

This precisely means that for the isomorphism  $\Phi$  we have that if  $X \in \prod_{\mathcal{F}} \text{GL}_n(R)$  and  $X - E \in (\prod_{\mathcal{F}} M_n(R))_g$ , then  $\Phi(X) - E \in (\prod_{\mathcal{F}} M_m(S))_g$ , i. e., the isomorphism  $\Phi$  respects grading. Similarly the inverse isomorphism also respects grading.

Therefore by Theorem 6 we get that there exist central idempotents

$$e \in \left( \prod_{\mathcal{F}} M_n(R) \right)_0 \text{ and } f \in \left( \prod_{\mathcal{F}} M_m(S) \right)_0,$$

a ring isomorphism

$$\theta_1 : e \prod_{\mathcal{F}} M_n(R) \rightarrow f \prod_{\mathcal{F}} M_m(S)$$

and a ring anti-isomorphism

$$\theta_2 : (1 - e) \prod_{\mathcal{F}} M_n(R) \rightarrow (1 - f) \prod_{\mathcal{F}} M_m(S),$$

preserving grading.

Since the rings  $R$  and  $S$  contain only finite number of central idempotents from the 0-component, then the matrix rings  $M_n(R)$  and  $M_m(S)$  also contain only finite number of central idempotents from the 0-component (all of them are of the form  $\alpha E$ , where  $\alpha$  is a central idempotent of the initial ring). Consequently the ultrapowers  $\prod_{\mathcal{F}} M_n(R)$  and  $\prod_{\mathcal{F}} M_m(S)$  also contain only finite number of central idempotents from the 0-component, which are all obtained from the idempotents of the initial rings.

Therefore elementary equivalence of linear groups in language respecting grading is equivalent to the existence of central idempotents  $e \in R_0$  and  $f \in S_0$  such that

$$M_n \left( \prod_{\mathcal{F}} eR \right) \cong_{\text{gr}} M_m \left( \prod_{\mathcal{F}} fS \right)$$

and

$$M_n \left( \prod_{\mathcal{F}} (1 - e)R \right) \cong_{\text{gr}} M_m \left( \prod_{\mathcal{F}} (1 - f)S \right)^{op}.$$

Since taking ultrapower and taking matrix ring of a given size are commuting operations, we can rewrite the last two conditions as

$$\begin{cases} \prod_{\mathcal{F}} M_n(eR) & \cong_{\text{gr}} \prod_{\mathcal{F}} M_m(fS), \\ \prod_{\mathcal{F}} M_n((1 - e)R) & \cong_{\text{gr}} \prod_{\mathcal{F}} M_m((1 - f)S)^{op}, \end{cases}$$

which is equivalent by Keisler–Shelah Isomorphism Theorem to the pair of conditions

$$\begin{cases} M_n(eR) & \equiv_{\text{gr}} M_m(fS), \\ M_n((1-e)R) & \equiv_{\text{gr}} M_m((1-f)S), \end{cases}$$

which was to be obtained.  $\square$

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(Received: July 28, 2016)

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